Buckling Analysis of Shear Deformable Plates by Boundary Element Method

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SUMMARY

In this paper, the derivation of boundary integral equations for the buckling analysis of shear deformable plate are developed. Plate buckling equations are written as a standard eigenvalue problem. The formulation is formed by coupling boundary element formulations of shear deformable plate and two dimensional plane stress elasticity. The eigenvalue problem of plate buckling yields a critical load factor and buckling modes. The domain integrals which appear in this formulation are treated in two different ways: initially the integrals are evaluated using constant cells, and next, they are transformed into equivalent boundary integrals using the dual reciprocity method (DRM). Several examples with different geometry, loading and boundary conditions are presented to demonstrate the accuracy of the formulation. Copy © 2004 John Wiley & Sons, Ltd.

KEY WORDS: boundary element method, shear deformable plate, buckling, eigenvalue

1. Introduction

Buckling analysis of compression panels are particularly important in aerospace structures. Phenomenon of the plate buckling has been investigated analytically and experimentally since the first experimental observation almost 150 years ago [1]. Analytical solutions of plate buckling based on the classical plate theory can be found in References [2, 3]. Numerical method such as the Finite Element Method (FEM) has been used by many researchers to investigate the problems [4, 5, 6]. Liu [7], applied the differential quadrature element method based on the Mindlin plate theory to the buckling analysis of discontinuous rectangular plates.

More recently, the boundary element method (BEM), has been applied to the field of plate buckling. Syngellakis and Elzein [8], extended the boundary element solution of the plate buckling based on Kirchhoff theory to accommodate any combination of loadings and support conditions. Nerantzaki and Katsidelakis [9], developed a BEM-based method for buckling of plates with variable thickness. Lin et al. [10], developed a more general boundary element formulation for wide variety of boundary conditions and arbitrary planar shapes to investigate the stability of elastic plate. Other works on elastic buckling analysis of plate using boundary element can also be found in [11, 12]. For stress concentration problems the shear deformable theory proposed by Reissner is generally preferred as the Kirchhoff theory is well known to lead to inaccurate results [13].
This paper presents the derivation of boundary integral equations for the buckling analysis of shear deformable plate. Plate buckling equations are written as a standard eigenvalue problem. The formulation is formed by coupling boundary element formulations of shear deformable plate and two dimensional plane stress elasticity. The domain integrals which appear in this formulation are treated in two different ways: initially the integrals are evaluated using constant cells, and next, they are transformed into equivalent boundary integrals using the dual reciprocity method (DRM). The eigenvalue problem of plate buckling yields a critical load factor and buckling modes. Several examples with different geometry, loading and boundary conditions are presented to demonstrate the accuracy of the formulation.

2. Governing Equations

The governing equations for plate buckling analysis can be written in a compact form using indicial notation as below:

\[ M_{\alpha\beta,\beta} - Q_{\alpha} = 0 \]  
(1)

\[ Q_{\alpha,\alpha} + (N_{\alpha\beta}w_{3,\beta}),\alpha + q = 0 \]  
(2)

\[ N_{\alpha\beta,\beta} = 0 \]  
(3)

where,

\[ M_{\alpha\beta} = \frac{1 - \nu}{2} D(w_{\alpha,\beta} + w_{\beta,\alpha} + \frac{2\nu}{1 - \nu} w_{\gamma,\gamma}\delta_{\alpha\beta}) \]  
(4)

\[ Q_{\alpha} = C(w_{\alpha} + w_{3,\alpha}) \]  
(5)

\[ N_{\alpha\beta} = \frac{1 - \nu}{2} B(u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1 - \nu} u_{\gamma,\gamma}\delta_{\alpha\beta}) \]  
(6)

\(B(= Eh/(1 - \nu^2))\) is known as the tension stiffness; \(D(= Eh^3/[12 (1 - \nu^2)])\) is the bending stiffness of the plate; \(C(=D(1 - \nu)\lambda^2)/2\) is the shear stiffness; \(\lambda(=\sqrt{10}/h)\) is shear factor; \(h\) is the thickness of the plate; \(\nu\) is Poisson’s ratio; \(N_{\alpha\beta}\) are stress resultants for two-dimensional plane stress elasticity; \(Q_{\alpha}\) and \(M_{\alpha\beta}\) are stress resultants plate bending problems; \(u_{\alpha}\) and \(w_{3}\) are translation of displacements in \(x_1, x_2\) (in-plane) and \(x_3\) (out of plane), \(w_{\alpha}\) are rotations in \(x_{\alpha}\) direction; and \(\delta_{\alpha\beta}\) is the Kronecker delta function.

Indicial notation is used throughout this paper. Greek indices will vary from 1 to 2 and Roman indices from 1 to 3.

3. Boundary Integral Equations

Here, the in-plane stress resultants in the domain due to external loads on the boundary is considered to be unknown. Therefore, determination of in-plane stress resultants in the domain is the first step in the solution of plate buckling. Next, the plate buckling equations are derived from the plate bending equations. Critical load factors are introduced into the equations as multiplication factors of body forces or transverse loads.
3.1. In-plane Stress Resultants.

In the absence of body forces, the boundary integral representation of in-plane displacement is given by

\[ C_{\theta \alpha}(x') u_{\alpha}(x') + \int_{\Gamma} T_{\theta \alpha}^*(x', x) u_{\alpha}(x) d\Gamma(x) = \int_{\Gamma} U_{\theta \alpha}^*(x', x) t_{\alpha}(x) d\Gamma(x) \]  
(7)

where \( C_{\theta \alpha}(x') \) are jump terms. The value of \( C_{\theta \alpha}(x') \) is equal to \( \frac{1}{2} \delta_{\theta \alpha} \) when \( x' \) is located on a smooth boundary.

The boundary integral of in-plane displacements in the domain can be expressed as

\[ u_{\alpha}(X') + \int_{\Gamma} T_{\theta \alpha}(X', x) u_{\alpha}(x) d\Gamma(x) = \int_{\Gamma} U_{\theta \alpha}(X', x) t_{\alpha}(x) d\Gamma(x) \]  
(8)

The in-plane stress resultants at domain point \( X' \) are written as

\[ N_{\alpha \beta}^{linear}(X') = \int_{\Gamma} U_{\alpha \beta}^*(X', x) t_{\Delta}(x) d\Gamma(x) - \int_{\Gamma} T_{\Delta \alpha \beta}^*(X', x) u_{\Delta}(x) d\Gamma(x) \]  
(9)

The fundamental solutions \( T_{\theta \alpha}^*, U_{\theta \alpha}^*, U_{\Delta \alpha \beta}^* \) and \( T_{\Delta \alpha \beta}^* \) are listed in Appendix A.

3.2. Plate Buckling Problem.

The plate bending equation is transformed into an equivalent plate buckling formulation by introducing critical load factor \( \lambda \) as follows:

\[ C_{ij}(x') w_i(x') + \int_{\Gamma} P_{ij}^*(x', x) w_j(x) d\Gamma(x) \]

\[ = \int_{\Gamma} W_{ij}^*(x', x) p_j(x) d\Gamma(x) + \lambda \int_{\Omega} W_{ij}^*(x', X) q(X) d\Omega(X) \]

\[ + \lambda \int_{\Omega} W_{ij}^*(x', X)(N_{\alpha \beta} w_{3,\beta})_{,\alpha}(X) d\Omega(X) \]  
(10)

where the terms \( C_{ij}(x') \) are equal to \( \frac{1}{2} \delta_{ij} \) when \( x' \) is located on a smooth boundary.

Expanding the last integral in Equation (10), gives:

\[ C_{ij} w_i(x') + \int_{\Gamma} P_{ij}^*(x', x) w_j(x) d\Gamma(x) \]

\[ = \int_{\Gamma} W_{ij}^*(x', x) p_j(x) d\Gamma(x) + \lambda \int_{\Omega} W_{ij}^*(x', X) q(X) d\Omega(X) \]

\[ + \lambda \int_{\Omega} W_{ij}^*(x', X)(N_{\alpha \beta} w_{3,\beta})_{,\alpha}(X) + N_{\alpha \beta} w_{3,\beta \alpha}(X) d\Omega(X) \]  
(11)

It is worth noting that equilibrium of in-plane resultant forces \( N_{\alpha \beta} \) is not implicitly satisfied by the two-dimensional BEM formulation. Hence it is important to retain the term \( N_{\alpha \beta,\alpha} \) in the plate bending equation. The deflection equation \( w_3 \) at the domain points \( X' \) is required as an additional equation to arrange an eigenvalue equation, as follows:
\[ w_3(X') = \int_{\Gamma} W_{3j}(X', x)p_j(x)\,d\Gamma(x) - \int_{\Gamma} P_{3j}^*(X', x)w_j(x)\,d\Gamma(x) \]
\[ + \lambda \int_{\Omega} W_{33}^*(X', X)q(X)\,d\Omega(X) \]
\[ + \lambda \int_{\Omega} W_{33}^*(X', X)(N_{\alpha\beta}\,w_{3,\beta} + N_{\alpha\beta}w_{3,\beta\alpha})(X)\,d\Omega(X) \quad (12) \]

To arrange an eigenvalue equation, the derivatives of \( w_{3,\beta}(X) \) and \( w_{3,\alpha\beta}(X) \) have to be expressed in terms of \( w_3(X) \). The \( w_{3,\beta}(X) \) and \( w_{3,\alpha\beta}(X) \) terms are approximated by a radial basis function \( f(r) \) as follows:

\[ w_3(x_1, x_2) = \sum_{m=1}^{M} f(r)^m \Psi^m \quad (13) \]

where \( M \) is the total number of selected points and \( f(r) = 1 + r^2 \) is the a radial basis function.

\[ r = \sqrt{(x_1 - x_1^m)^2 + (x_2 - x_2^m)^2} \quad (14) \]

The vector \( \Psi^m \) in Equation (13) is calculated as follows. By taking the values \( w_3(x_1, x_2) \) at \( M \) different points, a set of equations is obtained which may be expressed in a matrix form as

\[ w_3 = F\,\Psi \]

where each column of matrix \( F \) consists of a vector \( f_m \) containing the values of the function \( f^m(r) \) evaluated at the \( M \) collocation points. The \( \Psi^m \) coefficients are determined by inverting the matrix \( F \) as follows:

\[ \Psi = F^{-1}w_3 \quad (15) \]

Therefore, the first derivative of deflection \( w_{3,\beta} \) is expressed as

\[ w_{3,\beta}(x_1, x_2) = f(r)_{,\beta}(F^{-1}w_3) \quad (16) \]

and the second derivative of deflection \( w_{3,\alpha\beta} \) can be derived in a similar way as above

\[ \Psi = F^{-1}w_{3,\beta} \quad (17) \]

Therefore,

\[ w_{3,\alpha\beta}(x_1, x_2) = f(r)_{,\alpha}(F^{-1}w_{3,\beta}) \quad (18) \]

Substituting Equation (16) into Equation (18), gives

\[ w_{3,\alpha\beta}(x_1, x_2) = f(r)_{,\alpha}f(r)_{,\beta}F^{-1}w_3 \quad (19) \]

Similar to the above expressions, the derivative of in-plane stress resultants \( N_{\alpha\beta,\alpha} \) can be expressed as

\[ N_{\alpha\beta,\alpha}(x_1, x_2) = f(r)_{,\alpha}F^{-1}N_{\alpha\beta} \quad (20) \]
Equation (11) can be written as

\[
C_{ij}w_i(x') + \int_{\Gamma} P_{ij}^*(x', x)w_j(x)d\Gamma(x) \\
= \int_{\Gamma} W_{ij}^*(x', x)p_j(x)d\Gamma(x) \\
+ \lambda \int_{\Omega} W_{ij}^*(x', X)q(X)d\Omega(X) \\
+ \lambda \int_{\Omega} W_{ij}^*(x', X)(N_{\alpha\beta,\alpha}f(r),_{\beta}F^{-1}w_3)(X)d\Omega(X) \\
+ \lambda \int_{\Omega} W_{ij}^*(x', X)(N_{\alpha\beta}f(r),_{\alpha}f(r),_{\beta}F^{-1}w_3)(X)d\Omega(X)
\]

Equation (12) can also be written as

\[
w_3(X') = \int_{\Gamma} W_{3j}^*(X', x)p_j(x)d\Gamma(x) \\
- \int_{\Gamma} P_{3j}^*(X', x)w_j(x)d\Gamma(x) \\
+ \lambda \int_{\Omega} W_{33}^*(X', X)q(X)d\Omega(X) \\
+ \lambda \int_{\Omega} W_{33}^*(X', X)(N_{\alpha\beta}f(r),_{\alpha}f(r),_{\beta}F^{-1}w_3)(X)d\Omega(X) \\
+ \lambda \int_{\Omega} W_{33}^*(X', X)(N_{\alpha\beta}f(r),_{\alpha}f(r),_{\beta}F^{-1}w_3)(X)d\Omega(X)
\]

Equation (21) can be expressed as

\[
C_{ij}w_i(x') + \int_{\Gamma} P_{ij}^*(x', x)w_j(x)d\Gamma(x) \\
= \int_{\Gamma} W_{ij}^*(x', x)p_j(x)d\Gamma(x) + \lambda \int_{\Omega} W_{33}^*(x', X)f_b(X)d\Omega(X)
\]

where

\[
f_b = q + N_{\alpha\beta,\alpha}f(r),_{\beta}F^{-1}w_3 + N_{\alpha\beta}f(r),_{\alpha}f(r),_{\beta}F^{-1}w_3
\]

Thus, Equation (22) can also be written as

\[
w_3(X') = \int_{\Gamma} W_{3j}^*(X', x)p_j(x)d\Gamma(x) \\
- \int_{\Gamma} P_{3j}^*(X', x)w_j(x)d\Gamma(x) \\
+ \lambda \int_{\Omega} W_{33}^*(X', X)f_b(X)d\Omega(X)
\]

The kernel solutions \( P_{ij}^* \) and \( W_{ij}^* \) are listed in Appendix A.
4. Numerical Implementation

The quadratic isoparametric boundary elements are used to discretise along the boundary, while for the domain, the constant cell elements (as shown in Figure 1(a)) are used to describe the geometry.

Equation (23) can be rewritten in a discretised form as

\[
C_{ij} w_i(x') + \sum_{n=1}^{N_n} w_n^a \int_{\xi=-1}^{\xi=1} P_{ij}^*(x', x) M^a(\xi) J_n(\xi) d\xi \\
= \sum_{k=1}^{N_e} \sum_{\alpha=1}^{3} p_{ij}^{a\alpha} \int_{\xi=-1}^{\xi=1} W_{ij}^*(x', x) M^\alpha(\xi) J_n(\xi) d\xi \\
+ \lambda \sum_{n=1}^{N_e} f_b^k \int_{\eta=-1}^{\eta=1} \int_{\xi=-1}^{\xi=1} W_{ij}^*(x', X) J_k(\xi, \eta) d\xi d\eta
\]

where \( N_e \) and \( N_n \) are number of boundary elements and internal cells respectively; \( J_n(\xi) \) is the Jacobian of transformation for boundary elements and defined as follows:

\[
J_n(\xi) = \sqrt{\frac{\partial x_\theta(\xi)}{\partial \xi} \frac{\partial x_\theta(\xi)}{\partial \xi}}
\]

The quadratic shape functions \( M^\alpha \) are defined as follows:

\[
M^1(\xi) = \frac{1}{2} \xi(\xi - 1) \\
M^2(\xi) = (1 - \xi)(1 + \xi) \\
M^3(\xi) = \frac{1}{2} \xi(\xi + 1)
\]

The Jacobian of transformation \( J_k(\xi, \eta) \) for cell elements is defined as:

\[
J_k(\xi, \eta) = \sqrt{N^{2}_{31} + N^{2}_{32} + N^{2}_{33}}
\]
where $N_{ij}$ is a minor of
\[
\begin{bmatrix}
\frac{\partial x_1(\xi, \eta)}{\partial \xi} & \frac{\partial x_2(\xi, \eta)}{\partial \xi} & \frac{\partial x_3(\xi, \eta)}{\partial \xi} \\
\frac{\partial x_1(\eta)}{\partial \eta} & \frac{\partial x_2(\xi, \eta)}{\partial \eta} & \frac{\partial x_3(\xi, \eta)}{\partial \eta} \\
1 & 1 & 1
\end{bmatrix}
\] (30)

There is a weak singular term in the domain integral in Equation (23). When the integral are computed numerically using cell discretisation, the weak singular kernels are treated using a triangle to square transformation technique as explained in Reference [15].

In this case, Equation (23) can be expressed in matrix form as
\[
\mathbf{H}^p \mathbf{w} - \mathbf{G}^p \mathbf{p} = \lambda \mathbf{G}^p_3 \mathbf{f}_b
\] (31)
where $\mathbf{H}^p$ and $\mathbf{G}^p$ are boundary element influence matrices for plate bending, while $\mathbf{G}^p_3$ is the domain coefficient matrix.

5. Boundary Transformation of Domain Integral

In the boundary integral equation (23), there is a domain integral which can be written as follows:
\[
I_p = \int_{\Omega} W_{i3} f_b d\Omega
\] (32)

The dual reciprocity method for shear deformable plate developed by Wen et al.[16] can be used to evaluate the integral Equation (32). Assume that the term $f_b$ are the body forces, therefore they can be approximated by
\[
f_b = \sum_{m=1}^{M} f(r)^m \phi_j^m
\] (33)
where $f(r)$ is a radial basis function (see Appendix B), the $\phi_j^m$ are a set of unknown coefficients, $r$ is denoted as Equation (14), $M$ is the total number of the selected points (see Figure 1(b) for the DRM model).

The $\phi_j^m$ are coefficients which are determined by values at the selected points $M$ as follows
\[
\phi = \mathbf{F}^{-1} f_b
\] (34)

The boundary integral equation (23) can be written as
\[
C_{ij}(x') \hat{w}_{mj}(x') + \int_{\Gamma} P_{ij}^*(x', x) \hat{w}_{mj}(x) d\Gamma(x)
\]
\[
= \int_{\Gamma} W_{ij}^*(x', x') \hat{p}_{mj}(x) d\Gamma(x) + \int_{\Omega} W_{i3}^*(x', X) F_m(r) d\Omega(X)
\] (35)
The domain integral in Equation (32) can now be expressed in terms of boundary integrals as

\[
\int_{\Omega} W_{i3}^*(x',X)f_b(X)\,d\Omega(X) = \sum_{m=1}^{M} \phi_j^m(C_{ij}(x')\tilde{w}_{mj}(x') + \int_{\Gamma} P_{ij}^*(x',x)\tilde{w}_{mj}(x)\,d\Gamma(x)} - \int_{\Gamma} W_{ij}^*(x',x)\tilde{p}_{mj}(x)\,d\Gamma(x)
\] (36)

The boundary integral equation (23) can be written in a discretised form as

\[
C_{ij}w_i(x') + \sum_{n=1}^{N_e} \sum_{m=1}^{3} w_{jm}^p \int_{\xi=-1}^{\xi=1} P_{ij}^*(x',x)\Phi^m(\xi)J_n(\xi)d\xi = \sum_{k=1}^{N_e} \sum_{m=1}^{3} \Phi^m(\xi)J_n(\xi)d\xi + \lambda I^p
\] (37)

Equation (37) can be expressed in matrix form as

\[
H^p w - G^p p = \lambda(H^p \hat{w} - G^p \hat{p})F^{-1}f_b
\] (38)

where \(\hat{w}\) and \(\hat{p}\) are matrices of nodal values of particular solutions on the boundary, while \(H^p\) and \(G^p\) are the same as in Section 3.

6. Numerical Procedure

In this section, the numerical procedure for calculating the critical load of plates is presented. The first step is to solve boundary integral of in-plane problem and calculate the stress resultants at the domain points. Next the boundary integral formulation of buckling problem is solved.

6.1. Determination of the in-plane stresses.

After discretising and introducing boundary conditions into Equation (7), the system of algebraic equation can be arranged as

\[
[C]\{u\} + [H^s]\{u\} = [G^s]\{t\}
\] (39)

where coefficient matrices \(H^s\) and \(G^s\) are obtained from the fundamental solutions.

The known and unknown quantities in Equation (39) can be arranged as a set of linear algebraic equation:

\[
[A]\{X\} = \{F\}
\] (40)

where matrix \(X\) contains the unknown vectors of \(u\) and \(t\). The vector \(F\) is obtained by multiplying the related matrices of \(H^s\) or \(G^s\) by the known vectors of \(u\) or \(t\).

Once Equation (40) has been solved, in-plane stresses \(N_{11}, N_{12}\), and \(N_{22}\) in the domain (Equation 9) can be calculated. The stresses are required to solve the plate buckling problem.
6.2. Solving plate buckling problem.

Similar to the in-plane stresses procedure, after applying boundary conditions to Equation (23), it can be written as a system of algebraic equation:

\[
[C] \{w\} + [H]^p \{w\} = [G]^p \{p\} + \lambda[G]^p_{eq} \{f_b\} \tag{41}
\]

in which \(G^p_{eq} = G^p_3\) for domain cell discretisation, and \(G^p_{eq} = (H^p\hat{w} - G^p\hat{p})F^{-1}\) for the dual reciprocity method. The \(q(X)\) quantities in Equations (21) and (22) are set to zero. The term \(f_b(X)\) (Equation 24) can be expressed in term of \(w_3(X)\), as follows

\[
f_b(X) = f_{bw}(X)w_3(X) \tag{42}
\]

where \(f_{bw} = N_{linear}^{\alpha\beta\alpha}f(r)_{\beta}F^{-1} + N_{linear}^{\alpha\beta\alpha}f(r)_{\alpha}f(r)_{\beta}F^{-1}\).

Equation (41) can be rearranged in a similar manner as Equation (40), and give

\[
[B]_{3N \times 3N} \{Y\}_{3N} = \lambda [K]_{3N \times L} \{w_3\}_L \tag{43}
\]

\[
K = G^p_{eq}f_{bw} \tag{44}
\]

where the matrix \(B\) contains the coefficient matrices \(H^p\) and \(G^p\). \(N\) and \(L\) are number of boundary elements and domain points, respectively.

Equation (25) also can be written in matrix form as follows:

\[
[I] \{w_3\}_L = [BB]_{L \times 3N} \{Y\}_{3N} + \lambda [KK]_{L \times L} \{w_3\}_L \tag{45}
\]

where matrix \([I]\) is an identity matrix. The matrix \([BB]\) contains coefficient matrices related to the fundamental solutions of \(W^{*}_{3j}\) and \(P^{*}_{3j}\). The matrix \([KK]\) are obtained by multiplication of coefficient matrix related to the fundamental solution \(W^{*}_{33}\) and matrix \(f_{bw}\).

Equation (43) can be rearranged in term of unknown vector \(\{Y\}_{3N}\),

\[
\{Y\}_{3N} = \lambda [B]^{-1}_{3N \times 3N}[K]_{3N \times L} \{w_3\}_L \tag{46}
\]

where matrix \(B^{-1}\) is the inverse matrix of \(B\).

The substitution of Equation (46) into Equation (45) yields:

\[
[I] \{w_3\}_L = \lambda [BB]_{L \times 3N}[B]^{-1}_{3N \times 3N}[K]_{3N \times L} \{w_3\}_L
+ \lambda [KK]_{L \times L} \{w_3\}_L \tag{47}
\]

Then,

\[
[I] \{w_3\}_L = \lambda ([BB]_{L \times 3N}[B]^{-1}_{3N \times 3N}[K]_{3N \times L}
+ [KK]_{L \times L}) \{w_3\}_L \tag{48}
\]

Equation (48) can be written as a standard eigenvalue problem equation as follows:

\[
(\psi I - \frac{1}{\lambda}[I]) \{w_3\}_L = 0 \tag{49}
\]
Figure 2: Plate buckling model with different geometries, loadings and boundary conditions.

where \( \psi = [BB]_{L \times 3N}[B]^{-1}(3N \times 3N)[K]_{3N \times L} + [KK]_{L \times L} \).

Buckling analysis of shear deformable plates has been presented as a standard eigenvalue problem. Buckling modes correspond to the problem can be obtained by solving Equation (49).

7. Numerical Examples

Several numerical examples are presented to demonstrate the accuracy of the proposed method for analysis of plate buckling problems with different geometries, loadings and boundary conditions (see Figure 2). The BEM results are compared with analytical and finite element results. In the following examples, the buckling coefficient \( K \) is defined by

\[
K = \frac{b^2}{\pi^2 D} T
\]

where \( T \) is critical compression load \( \sigma_{cr} \) or critical shear load \( \tau_{cr} \), \( b \) is the width or diameter of plates.

7.1. Convergence study of simply supported square plate subjected to compression loads.

In this example, a square plate as shown in Figure 3 is analysed. Six different BEM meshes of domain cells and domain points are used. Convergence study of simply supported square plate is performed and the buckling coefficients \( K \) are compared with analytical result [3].

The number of boundary elements and cell elements are shown in Table 1, and the number of domain points in Table 2. It can be seen, convergence of the results is achieved with increasing number of cells and domain points. It can also be seen that the BEM results are in good agreement with analytical results.
Table 1: Buckling coefficients using domain cells.

<table>
<thead>
<tr>
<th>No</th>
<th>Boundary</th>
<th>Domain</th>
<th>K BEM</th>
<th>K analytical</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20 elems</td>
<td>5 x 5 cells</td>
<td>4.241</td>
<td>4.000</td>
<td>6.030</td>
</tr>
<tr>
<td>2</td>
<td>24 elems</td>
<td>6 x 6 cells</td>
<td>4.173</td>
<td>4.000</td>
<td>4.325</td>
</tr>
<tr>
<td>3</td>
<td>28 elems</td>
<td>7 x 7 cells</td>
<td>4.143</td>
<td>4.000</td>
<td>3.575</td>
</tr>
<tr>
<td>4</td>
<td>32 elems</td>
<td>8 x 8 cells</td>
<td>4.079</td>
<td>4.000</td>
<td>1.975</td>
</tr>
<tr>
<td>5</td>
<td>36 elems</td>
<td>9 x 9 cells</td>
<td>4.068</td>
<td>4.000</td>
<td>1.700</td>
</tr>
<tr>
<td>6</td>
<td>40 elems</td>
<td>10 x 10 cells</td>
<td>4.041</td>
<td>4.000</td>
<td>1.025</td>
</tr>
</tbody>
</table>

Table 2: Buckling coefficients using the dual reciprocity technique.

<table>
<thead>
<tr>
<th>No</th>
<th>Boundary</th>
<th>Domain</th>
<th>K BEM</th>
<th>K analytical</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32 elems</td>
<td>5 x 5 points</td>
<td>4.189</td>
<td>4.000</td>
<td>4.720</td>
</tr>
<tr>
<td>2</td>
<td>32 elems</td>
<td>6 x 6 points</td>
<td>4.141</td>
<td>4.000</td>
<td>3.515</td>
</tr>
<tr>
<td>3</td>
<td>32 elems</td>
<td>7 x 7 points</td>
<td>4.060</td>
<td>4.000</td>
<td>1.510</td>
</tr>
<tr>
<td>4</td>
<td>32 elems</td>
<td>8 x 8 points</td>
<td>4.032</td>
<td>4.000</td>
<td>0.808</td>
</tr>
<tr>
<td>5</td>
<td>32 elems</td>
<td>9 x 9 points</td>
<td>3.985</td>
<td>4.000</td>
<td>0.387</td>
</tr>
<tr>
<td>6</td>
<td>32 elems</td>
<td>10 x 10 points</td>
<td>3.999</td>
<td>4.000</td>
<td>0.025</td>
</tr>
</tbody>
</table>

Figure 3: Square plate subjected to compression loads
Figure 4: Square and circular buckling models.

Table 3: Buckling coefficients of square and circular plates.

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Domain Cell</th>
<th>DRM</th>
<th>FEM</th>
<th>Analytical</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-A</td>
<td>10.387</td>
<td>10.142</td>
<td>10.392</td>
<td>10.070</td>
</tr>
<tr>
<td>I-B</td>
<td>4.041</td>
<td>3.999</td>
<td>4.011</td>
<td>4.000</td>
</tr>
<tr>
<td>I-C</td>
<td>7.757</td>
<td>7.683</td>
<td>7.796</td>
<td>7.690</td>
</tr>
<tr>
<td>I-D</td>
<td>6.972</td>
<td>6.781</td>
<td>6.882</td>
<td>6.740</td>
</tr>
<tr>
<td>I-E</td>
<td>1.724</td>
<td>1.712</td>
<td>1.718</td>
<td>1.700</td>
</tr>
<tr>
<td>I-F</td>
<td>1.417</td>
<td>1.428</td>
<td>1.422</td>
<td>1.440</td>
</tr>
<tr>
<td>II-A</td>
<td>5.779</td>
<td>5.889</td>
<td>5.921</td>
<td>5.910</td>
</tr>
<tr>
<td>II-B</td>
<td>1.661</td>
<td>1.689</td>
<td>1.714</td>
<td>1.702</td>
</tr>
</tbody>
</table>

I : square plate
II : circular plate
A : sides and ends clamped
B : sides and ends simply supported
C : sides clamped, ends simply supported
D : ends clamped, sides simply supported
E : one side free, one side clamped, end simply supported
F : one side free, the other side and ends simply supported

7.2. Square and circular plates subjected to compression loads with different boundary conditions.

Examples of buckling problems for square and circular plates subjected to compression loads with different boundary conditions are presented. The models are shown in Figure 4. Initially, the square plate is discretised into 40 boundary elements and 100 domain cells, and next analysed using the dual reciprocity technique with 32 boundary elements and 100 domain points. The circular plate is discretised using 16 boundary elements and 49 domain cells. It is also analysed using the dual reciprocity technique with 16 boundary elements and 32 domain points. Table 3 presents the BEM results of the examples as well as analytical and finite element results. As it can be seen from Table 3 that BEM results are in good agreement (the maximum error is less than 4%) with both analytical and finite element results.
7.3. Rectangular plate subjected to compression load with different boundary conditions.

In this example, a rectangular plate (as in Figure 5) subjected to compression loads with different boundary conditions is presented. The buckling coefficients $K$ for different aspect ratio $a/b$ are presented in Figure 6 and the accuracy achieved by BEM can be considered satisfactory. The legends in the Figure 5 stand for boundary conditions as follows:

- cccc : sides and ends clamped
- ssss : sides and ends simply supported
- cscs : ends clamped, sides simply supported
- sfsc : one side free, one side clamped, end simply supported
- ssff : one side free, the other side and ends simply supported

The buckling contours and modes of simply supported rectangular plate are presented in Figures 7 and 8 respectively. From Figure 8, it can be seen that increasing the aspect ratio $a/b$ will increase a number of half-waves. Contour plot presented in Figure 7 are related to the buckling modes.

7.4. Rectangular plate subjected to shear loads.

In this example, shear buckling of rectangular plate with different aspect ratio $a/b$ is presented. The buckling model is shown in Figure 9. Boundary conditions applied are simply supported and clamped. The results are plotted in Figure 10 and are compared with analytical and finite element results. Good agreements ($< 1.5\%$ error) is achieved in both cases. Buckling contours and modes of simply supported rectangular plate are also presented. The contour types of buckling is shown in Figure 11. The shear buckling modes of simply supported rectangular plate can be seen in Figure 12.

7.5. Secondary buckling of simply supported rectangular plate subjected to compression loads.

In this example, secondary buckling mode of simply supported rectangular plate subjected to compression loads is investigated. The primary buckling mode of the example is shown in Figure 14. The results are plotted with different aspect ratio and are compared with analytical results. The results in Figure 13 are considered satisfactory. Secondary buckling modes are plotted in Figure 15.
Figure 6: Buckling coefficients of rectangular plate with different boundary conditions.

Figure 7: Contour plot of simply supply supported rectangular plate buckling
Figure 8: Buckling modes of simply supported rectangular plate

Figure 9: Rectangular plate subjected to shear loads.
Figure 10: Shear buckling coefficient of rectangular plate.

Figure 11: Contour plot of shear buckling of simply supported rectangular plate.
Figure 12: Shear buckling modes of simply supported rectangular plate.

Figure 13: Secondary buckling coefficients of simply supported rectangular plate under compression loads.
Figure 14: Primary buckling of simply supported rectangular plate under compression loads.

Figure 15: Secondary buckling mode of simply supported rectangular plate subjected to compression loads.
In this paper, the boundary element method formulation of buckling analysis of shear deformable plates was presented. Plate buckling equations were presented as a standard eigenvalue problem. The domain integrals which appear in this formulation are treated in two different ways: initially the integrals are evaluated using constant cells, and next, the dual reciprocity method (DRM) was used to transform the domain integrals into equivalent boundary integrals using \( r \). The eigenvalue problem of plate buckling yields a critical load factor and buckling modes.

Several examples of plates buckling with different geometries, loadings and boundary conditions were presented. The results presented are in good agreement with analytical and finite element results. The following conclusions can be made based on the results presented.

- The boundary integral equation can be used as an effective tool to solve buckling problems of plates with different geometries, loadings and boundary conditions.
- The evaluation of domain integrals using the dual reciprocity method were found to be more accurate than domain integration using constant cell discretization. It is anticipated that higher order cells would be more accurate, however they will require more elaborate integration schemes to deal with strongly singular integrals [14]. Mesh generation for the dual reciprocity method is much easier than domain integral method.
- The BEM results were shown to be in good agreement (mostly within 2%) with analytical and finite element results.

Appendix A: Fundamental solutions

Plate Bending Problem

The expressions for the kernels \( W_{ij}^* \) and \( P_{ij}^* \) are given by Vander Weeën [17] as follows:

\[
W_{\alpha\beta}^* = \frac{1}{8\pi D (1 - \nu)} \left\{ \left[ 8 B(z) - (1 - \nu) (2 \ln z - 1) \right] \delta_{\alpha\beta} \right. \\
- \left. \left[ 8 A(z) + 2(1 - \nu) \right] r_{,\alpha} r_{,\beta} \right\} \\
W_{\alpha\beta}^* = -W_{3\alpha}^* = \frac{1}{8\pi D (2 \ln z - 1)} r_{,\alpha} \\
W_{33}^* = \frac{1}{8\pi D (1 - \nu)} \lambda^2 \left[ (1 - \nu) z^2 (\ln z - 1) - 8 \ln z \right]
\] (A1)

and

\[
P_{\gamma\alpha}^* = -\frac{1}{4\pi r} \left\{ (4 A(z) + 2z K_1(z) + 1 - \nu) (\delta_{\alpha\gamma} r_{,n} + r_{,\alpha} n_{,\gamma}) \right. \\
+ \left. (4 A(z) + 1 + \nu) r_{,\gamma} n_{,\alpha} - 2(8 A(z) + 2z K_1(z) + 1 - \nu) r_{,\alpha} r_{,\gamma} n_{,n} \right\} \\
P_{\gamma 3}^* = \frac{\lambda^2}{2\pi} [B(z) n_{,\gamma} - A(z) r_{,\gamma} r_{,n}] \\
P_{3\alpha}^* = -\frac{(1 - \nu)}{8\pi} \left[ \left( \frac{1 + \nu}{1 - \nu} \ln z - 1 \right) n_{,\alpha} + 2 r_{,\alpha} r_{,n} \right] \\
P_{33}^* = -\frac{1}{2\pi r} r_{,n}
\] (A2)
The expression of $W_{ijk}^\star$, $P_{ijk}^\star$ and $Q_{ij}^\star$ are [17]:

\[ W_{\alpha\beta\gamma}^\star = \frac{1}{4\pi r} [(4A(z) + 2zK_1(z) + 1 - \nu)(\delta_{\beta\gamma}r,_{\alpha} + \delta_{\alpha\gamma}r,_{\beta}) \]
\[ - 2(8A(z) + 2zK_1(z) + 1 - \nu)r,_{\alpha}r,_{\beta}\gamma + (4A(z) + 1 + \nu)\delta_{\alpha\beta}r,_{\gamma}] \]
\[ W_{\alpha\beta3}^\star = -\frac{(1 - \nu)}{8\pi} \left[ \frac{(1 + \nu)}{(1 - \nu)} \ln z - 1 \right] \delta_{\alpha\beta} + 2r,_{\alpha}r,_{\beta} \]
\[ W_{3\beta\gamma}^\star = \frac{\lambda^2}{2\pi} [B(z)\delta_{\gamma\beta} - A(z)r,_{\gamma}r,_{\beta}] \]
\[ W_{3\beta3}^\star = \frac{1}{2\pi r} r,_{\beta} \]

(3A)

\[ P_{\alpha\beta\gamma}^\star = \frac{D(1 - \nu)}{4\pi r^2} [(4A(z) + 2zK_1(z) + 1 - \nu)(\delta_{\gamma\alpha}n,_{\beta} + \delta_{\gamma\beta}n,_{\alpha}) \]
\[ + (4A(z) + 1 + 3\nu)\delta_{\alpha\beta}n,_{\gamma} - (16A(z) + 6zK_1(z) + z^2K_0(z) + 2 - 2\nu) \]
\[ \times [(n,_{\alpha}r,_{\beta} + n,_{\beta}r,_{\alpha})r,_{\gamma} + (\delta_{\gamma\alpha}r,_{\beta} + \delta_{\gamma\beta}r,_{\alpha})n,_{\alpha}] \]
\[ - 2(8A(z) + 2zK_1(z) + 1 + \nu)(\delta_{\alpha\beta}r,_{\gamma}n,_{n} + n,_{\gamma}r,_{\alpha}r,_{n}) \]
\[ + 4(24A(z) + 8zK_1(z) + z^2K_0(z) + 2 - 2\nu)r,_{\alpha}r,_{\beta}r,_{\gamma}n,_{n}] \]
\[ P_{\alpha\beta3}^\star = \frac{-D(1 - \nu)}{4\pi r^2} [(2A(z) + zK_1(z))(r,_{\beta}n,_{\alpha} + r,_{\alpha}n,_{\beta}) \]
\[ - 2(4A(z) + zK_1(z))r,_{\alpha}r,_{\beta}n,_{n} + 2A(z)\delta_{\alpha\beta}n,_{n}] \]
\[ P_{3\beta\gamma}^\star = \frac{-D(1 - \nu)}{4\pi r^2} [(2A(z) + zK_1(z))(\delta_{\gamma\beta}r,_{n} + r,_{\gamma}n,_{\beta}) \]
\[ + 2A(z)n,_{\gamma}r,_{\beta} - 2(4A(z) + zK_1(z))r,_{\gamma}r,_{\beta}n,_{n}] \]
\[ P_{3\beta3}^\star = \frac{D(1 - \nu)}{4\pi r^2} [(z^2B(z) + 1)n,_{\beta} - (z^2A(z) + 2)r,_{\beta}r,_{n}] \]

(4A)

\[ Q_{\alpha\beta}^\star = \frac{-r}{64\pi} [(4\ln z - 3)[(1 - \nu)(r,_{\beta}n,_{\alpha} + r,_{\alpha}n,_{\beta}) + (1 + 3\nu)\delta_{\alpha\beta}n,_{n}] \]
\[ + 4[(1 - \nu)r,_{\alpha}r,_{\beta} + \nu\delta_{\alpha\beta}]n,_{n}] \]
\[ Q_{3\beta}^\star = \frac{1}{8\pi} [(2\ln z - 1)n,_{\beta} + 2r,_{\beta}r,_{n}] \]

(A5)

where

\[ A(z) = K_0(z) + \frac{2}{z} \left[ K_1(z) - \frac{1}{z} \right] \]
\[ B(z) = K_0(z) + \frac{1}{z} \left[ K_1(z) - \frac{1}{z} \right] \]

(6A)

in which $K_0(z)$ and $K_1(z)$ are modified Bessel functions of the second kind [18], $z = \lambda r$, $\lambda$ is the shear factor defined in section 2, $r$ is the absolute distance between the source and the field points, $r,_{\alpha} = r,_{\alpha}/r$, where $r,_{\alpha} = x,_{\alpha}(x') - x,_{\alpha}(x')$ and $n,_{n} = r,_{\alpha}n,_{\alpha}$.
Expanding the modified Bessel functions for small arguments:

\[ K_0(z) = \left[ -\gamma - \ln\left(\frac{z}{2}\right) \right] + \left[ -\gamma + 1 - \ln\left(\frac{z}{2}\right) \right] \frac{(z^2/4)}{(1!)}^2 + \left[ -\gamma + 1 + \frac{1}{2} - \ln\left(\frac{z}{2}\right) \right] \frac{(z^2/4^2)}{(2!)}^2 + \cdots \tag{A7} \]

\[ K_1(z) = \frac{1}{2} - \left[ -\gamma + 1 - \ln\left(\frac{z}{2}\right) \right] \frac{(z^2/4)^{1/2}}{0!1!} + \left[ -\gamma + 1 + \frac{1}{4} - \ln\left(\frac{z}{2}\right) \right] \frac{(z^2/4^3/2)}{1!2!} + \left[ -\gamma + 1 + \frac{1}{6} - \ln\left(\frac{z}{2}\right) \right] \frac{(z^2/4^5/2)}{2!3!} + \cdots \tag{A8} \]

where \( \gamma = 0.5772156649 \) is the Euler constant. Substitute Equations (A7) and (A8) into (A6) and take the limit as \( r \to 0 \):

\[ \lim_{r \to 0} A(z) = -\frac{1}{2}, \]

\[ \lim_{r \to 0} B(z) = -\frac{1}{2} \left[ \lim_{r \to 0} \ln\left(\frac{z}{2}\right) + \gamma + \frac{1}{2} \right] \tag{A9} \]

As it can be seen, \( A(z) \) is a smooth function, whereas, \( B(z) \) is a weakly singular \( O(\ln r) \). Therefore \( W^*_{ij} \) is weakly singular and \( P^*_{ij} \) has a strong (Cauchy principal value) singularity \( O(1/r) \).

In this work, the modified Bessel functions are evaluated using polynomial approximations given by Abramowitz and Stegun [18].

**Two-dimensional Plane Stress Problem**

The expressions for the kernels \( U^*_{\theta\alpha} \) and \( T^*_{\theta\alpha} \) are well known (Kelvin solution) for two-dimensional plane stress problems, and are given as:

\[ U^*_{\theta\alpha} = \frac{1}{4\pi B (1 - \nu)} \left[ (3 - \nu) \ln \left(\frac{1}{r}\right) \delta_{\theta\alpha} + (1 + \nu) r_{\theta\alpha} \right] \tag{A10} \]

\[ T^*_{\theta\alpha} = -\frac{1}{4\pi r} \left[ r_{\alpha} \right] (1 - \nu) \delta_{\theta\alpha} + 2(1 + \nu) r_{\theta\alpha} \] 
\[ + (1 - \nu) [r_{\theta\alpha} - n_{\alpha} r_{\theta}] \] 

where \( U^*_{\theta\alpha} \) are weakly singular kernels of order \( O(\ln \frac{1}{r}) \) and \( T^*_{\theta\alpha} \) are strongly singular in order \( O(1/r) \).

The expressions for the kernels \( U^*_{\alpha\beta\gamma} \) and \( T^*_{\alpha\beta\gamma} \) are:

\[ U^*_{\alpha\beta\gamma} = \frac{1}{4\pi r} \left[ (1 - \nu) (\delta_{\gamma\alpha} r_{\beta} + \delta_{\gamma\beta} r_{\alpha} - \delta_{\alpha\beta} r_{\gamma}) + 2 (1 + \nu) r_{\alpha} r_{\beta} r_{\gamma} \right] \tag{A12} \]
\[
T_{\alpha\beta\gamma}^* = \frac{B(1-\nu)}{4\pi r^2} \{2r_n [(1-\nu)\delta_{\alpha\beta}r_{,\gamma} + \nu(\delta_{\gamma\alpha}r_{,\beta} + \delta_{\gamma\beta}r_{,\alpha}) - 4(1+\nu)r_{,\alpha}r_{,\beta}r_{,\gamma}] \\
+ 2\nu(n_\alpha r_{,\beta}r_{,\gamma} + n_\beta r_{,\alpha}r_{,\gamma}) + (1-\nu)(2n_\gamma r_{,\alpha}r_{,\beta} + n_\beta \delta_{\alpha\gamma} + n_\alpha \delta_{\beta\gamma}) \\
- (1-3\nu)n_\gamma \delta_{\alpha\beta}\} \tag{A13}
\]

Appendix B: Particular solutions

Particular solutions derived by Wen, Aliabadi and Young [? are used for the dual reciprocity technique in the thesis and are given in the following sections.

**Particular solutions for plate bending**

Governing equation for shear deformable plate bending problem can be written as

\[
\dot{\mathbf{w}} = \mathbf{H} \mathbf{e} \varphi \tag{B1}
\]

where particular solutions of displacement \(\dot{\mathbf{w}} = \{\dot{w}_1, \dot{w}_2, \dot{w}_3\}^\top\), \(\mathbf{e} = \{e_1, e_2, e_3\}^\top\) is arbitrary constant vector and components of matrix \(\mathbf{H}\) are

\[
H_{\alpha\beta} = 2\delta_{\alpha\beta} \nabla^4 - [(1+\nu)\nabla^2 + (1-\nu)\lambda^2] \frac{\partial^2}{\partial x_\alpha \partial x_\beta}
\]

\[
H_{3\alpha} = -H_{\alpha 3} = -(1-\nu)(\nabla^2 - \lambda^2) \frac{\partial}{\partial x_\alpha}
\]

\[
H_{33} = (\nabla^2 - \lambda^2)[2\nabla^2 - (1-\nu)\lambda^2]/\lambda^2 \tag{B2}
\]

The function \(\varphi\) can be defined from Equation (50) such that

\[
D(1-\nu)(\nabla^2 - \lambda^2) \nabla^4 \varphi + F(r) = 0 \tag{B3}
\]

If \(e_1 = 0, e_2 = 0\) and \(e_3 = 1\), the particular solution used in Equation (35) can be written as

\[
\dot{w}_{m\alpha} = -\frac{1}{D} \frac{\partial \psi}{\partial x_\alpha}
\]

\[
\dot{w}_{m3} = \frac{1}{(1-\nu)D\lambda^2} [2\nabla^2 \psi - (1-\nu)\lambda^2 \psi] \tag{B4}
\]

where

\[
\nabla^4 \psi(r) + F(r) = 0 \tag{B5}
\]

The particular solutions of moment and shear force can be determined from the stress resultant-displacement relationship for shear deformable plate bending. The tractions on the boundary can be obtained by

\[
\dot{p}_{m\alpha} = \dot{M}_{\alpha\beta} n_\beta, \quad \dot{p}_{m3} = \dot{Q}_\alpha n_\alpha \tag{B6}
\]

If radial basis function \(F(r) = 1 + r\), The function \(\psi(r)\) can be solved from Equation (B5)

\[
\psi(r) = -\left(\frac{r^4}{64} + \frac{r^5}{225}\right) \tag{B7}
\]
and the rotations and deflection can be deduced

\[
\hat{w}_{m1}^3 = -\left(\frac{1}{16} + \frac{r}{45}\right) \frac{x_1 r^2}{D}
\]
\[
\hat{w}_{m2}^3 = -\left(\frac{1}{16} + \frac{r}{45}\right) \frac{x_2 r^2}{D}
\]
\[
\hat{w}_{m3}^3 = -\left(\frac{1}{2} + \frac{2r}{9}\right) \frac{r^2}{(1 - \nu)\lambda^2 D} + \left(\frac{1}{64} + \frac{r}{225}\right) \frac{1}{D}
\]

(B8)

The particular solutions of moments \(\hat{M}_{\alpha\beta}\) and shear forces \(\hat{Q}_\beta\) can be determined by the stress resultant-displacement relationship for shear deformable plate bending to give

\[
\hat{M}_{m11}^3 = -\left[\left(1 + \nu\right) \left(\frac{1}{8} + \frac{r}{15}\right) \left(x_1^2 + \nu x_2^2\right) + \left(1 + \nu\right) \left(\frac{r^2}{16} + \frac{r^3}{45}\right)\right]
\]
\[
\hat{M}_{m12}^3 = -\left(1 + \nu\right) \left(\frac{1}{8} + \frac{r}{15}\right) (x_1 x_2)
\]
\[
\hat{M}_{m22}^3 = -\left[\left(1 + \nu\right) \left(\nu x_1^2 + x_2^2\right) + \left(1 + \nu\right) \left(\frac{r^2}{16} + \frac{r^3}{45}\right)\right]
\]

(B9)

\[
\hat{Q}_{m1}^3 = -\frac{x_1}{2} \left(1 + \frac{2r}{3}\right)
\]
\[
\hat{Q}_{m2}^3 = -\frac{x_2}{2} \left(1 + \frac{2r}{3}\right)
\]

and the tractions on the boundary can be obtained from relationships in Equation (B6).

For the derivative of function \(F_{,\alpha} = x_{\alpha}/r\), the solution \(\psi^\alpha(r)\) can be found

\[
\psi^\alpha(r) = -\frac{r^3 x_{\alpha}}{45}
\]

(B10)

and particular solutions \(\hat{w}_{mk}^\alpha\) are

\[
\hat{w}_{m1}^1 = -(3x_1^2 + r^2) \frac{r}{45D}
\]
\[
\hat{w}_{m2}^1 = -\frac{x_1 x_2 r}{15D}
\]
\[
\hat{w}_{m3}^1 = -\left[30 - (1 - \nu)\lambda^2 r^2\right] \frac{r x_1}{45(1 - \nu)\lambda^2 D}
\]

(B11)

and the particular solutions of moments \(\hat{M}_{\alpha\beta}\) and shear forces \(\hat{Q}_\beta\) are

\[
\hat{M}_{m11}^1 = -\frac{x_1}{15} \left[\nu \left(\frac{x_1^2}{r} + 3r\right) + \left(\frac{x_2^2}{r} + r\right)\right]
\]
\[
\hat{M}_{m12}^1 = -\left(1 - \nu\right) \frac{x_2}{15} \left(\frac{x_1^2}{r} + r\right)
\]
\[ M_{m12}^{1} = -\frac{x_1}{15} \left[ \nu \left( \frac{x_1^2}{r} + 3r \right) + \left( \frac{x_2^2}{r} + r \right) \right] \]  
\[ \dot{Q}_{m1}^{1} = -\frac{1}{3} \left( \frac{x_1^2}{r} + r \right) \]  
\[ \dot{Q}_{m2}^{1} = -\frac{1}{3} \frac{x_1x_2}{r} \]  
for \( \alpha = 1 \), and

\[ \hat{w}_{m1}^{2} = -\frac{x_1x_2r}{15D} \]  
\[ \hat{w}_{m1}^{2} = -(3x_1^2 + r^2) \frac{r}{45D} \]  
\[ \hat{w}_{m3}^{2} = -[30 - (1 - \nu)\lambda^2 r^2] \frac{r}{45(1 - \nu)\lambda^2 D} \]  
and the particular solutions of moments \( \dot{M}_{\alpha\beta} \) and shear forces \( \dot{Q}_{\beta} \) are

\[ \dot{M}_{m11}^{2} = -\frac{x_2}{15} \left[ \nu \left( \frac{x_1^2}{r} + r \right) + \left( \frac{x_2^2}{r} + 3r \right) \right] \]  
\[ \dot{M}_{m12}^{2} = -(1 - \nu) \frac{x_1}{15} \left( \frac{x_1^2}{r} + r \right) \]  
\[ \dot{M}_{m22}^{2} = -\frac{x_2}{15} \left[ \nu \left( \frac{x_1^2}{r} + r \right) + \left( \frac{x_2^2}{r} + 3r \right) \right] \]  
\[ \dot{Q}_{m1}^{2} = -\frac{1}{3} \frac{x_1x_2}{r} \]  
\[ \dot{Q}_{m2}^{2} = -\frac{1}{3} \left( \frac{x_2^2}{r} + r \right) \]  
for \( \alpha = 2 \).

**Particular solutions for two-dimensional plane stress**

An expression displacement particular solution \( \hat{u}_{m\alpha}^{\gamma} \) can be found in polar coordinates with the use of the Galerkin vector \( G_{\alpha\beta} \) as

\[ \hat{u}_{m\alpha}^{\gamma}(r) = G_{\beta\alpha,\gamma\gamma}(r) - \frac{1 + \nu}{2} G_{\gamma\alpha,\beta\gamma}(r) \]  
where \( G_{\alpha\beta} \) satisfies

\[ \nabla^4 G_{\beta\alpha} + \frac{2}{(1 - \nu)B} \frac{x_\gamma}{r} \delta_{\gamma\beta} = 0 \]  
and a solution is determined by

\[ G_{\beta\alpha} = -\frac{r^3x_\gamma}{45(1 - \nu)B} \delta_{\alpha\beta} \]
Substituting Equation (B17) into Equation (B15), then the displacement particular solutions can be arranged as

\[ \hat{u}^{1}_{m1} = -\frac{2}{1-\nu} B \left[ \frac{rx_1}{3} - \frac{1+\nu}{30} \left( \frac{x_1^3}{r} + 3x_1r \right) \right] \]

\[ \hat{u}^{1}_{m2} = \frac{(1+\nu)}{15(1-\nu)B} \left( \frac{x_1^2x_2}{r} + x_2r \right) \]

and using strain displacement relationships for two-dimensional plane stress, the strain are obtained as

\[ \hat{\varepsilon}^{1}_{m11} = -\frac{2}{(1-\nu)} \left[ \left( \frac{x_1^2}{r} + \frac{r}{3} \right) - \frac{1+\nu}{30} \left( \frac{x_1^3}{r^3} + \frac{6x_1^2}{r} + 3r \right) \right] \]

\[ \hat{\varepsilon}^{1}_{m12} = -\frac{2}{(1-\nu)} \left[ \frac{x_1x_2}{6r} - \frac{1+\nu}{30} \left( \frac{x_1^3}{r^3} + \frac{3x_1x_2}{r} \right) \right] \]

\[ \hat{\varepsilon}^{1}_{m22} = \frac{2}{(1-\nu)} \frac{1+\nu}{30} \left( \frac{x_1^2x_2^2}{r^3} + 2r \right) \]

(B18)

The particular solution for membrane stress resultant can be derived by substituting Equation (B19) into the stress resultant-strain relationships for two-dimensional plane stress to give:

\[ \hat{N}^{1}_{m11} = B \left[ (1-\nu) \hat{\varepsilon}^{1}_{m11} + \nu \hat{\varepsilon}^{1}_{m12} \right] \]

\[ \hat{N}^{1}_{m12} = B(1-\nu) \hat{\varepsilon}^{1}_{m12} \]

\[ \hat{N}^{1}_{m22} = B \left[ (1-\nu) \hat{\varepsilon}^{1}_{m22} + \nu \hat{\varepsilon}^{1}_{m12} \right] \]

(B19)

and the traction particular solutions are obtained from

\[ \hat{t}^{1}_{\alpha\beta} = \hat{N}^{1}_{\alpha\beta\gamma} n_{\gamma} \]

(B20)

In the same way, displacement particular solutions \( \hat{u}^{2}_{m\alpha} \) can be obtained as follows:

\[ \hat{u}^{2}_{m1} = \frac{(1+\nu)}{15(1-\nu)B} \left( \frac{x_1^2x_1}{r} + x_1r \right) \]

\[ \hat{u}^{2}_{m2} = -\frac{2}{(1-\nu)B} \left[ \frac{rx_2}{3} - \frac{1+\nu}{30} \left( \frac{x_2^3}{r} + 3x_2r \right) \right] \]

and the strains are

\[ \hat{\varepsilon}^{2}_{m11} = \frac{2}{(1-\nu)} \frac{1+\nu}{30} \left( \frac{x_1^2x_2^2}{r^3} + 2r \right) \]

\[ \hat{\varepsilon}^{2}_{m12} = -\frac{2}{(1-\nu)} \left[ \frac{x_1x_2}{6r} - \frac{1+\nu}{30} \left( \frac{x_1^3}{r^3} + \frac{3x_1x_2}{r} \right) \right] \]

\[ \hat{\varepsilon}^{2}_{m22} = -\frac{2}{(1-\nu)} \left[ \left( \frac{x_2^2}{r} + \frac{r}{3} \right) - \frac{1+\nu}{30} \left( \frac{x_2^4}{r^3} + \frac{6x_2^2}{r} + 3r \right) \right] \]

(B21)
The particular solution for membrane stress resultant are

\[
\hat{N}_{m11}^2 = B \left[ (1 - \nu)\hat{\varepsilon}_{m11}^2 + \nu\hat{\varepsilon}_{maa}^2 \right]
\]

\[
\hat{N}_{m12}^2 = B(1 - \nu)\hat{\varepsilon}_{m12}^2
\]

\[
\hat{N}_{m22}^2 = B \left[ (1 - \nu)\hat{\varepsilon}_{m22}^2 + \nu\hat{\varepsilon}_{maa}^2 \right]
\]

and finally the traction particular solutions are obtained from

\[
t_{ma}^2 = \hat{N}_{maa}^2 n_\beta
\]  

\[(B24)\]

\[(B25)\]

**REFERENCES**


